

EIGENVALUES AND LAMBDA CONSTANTS ON RIEMANNIAN SUBMERSIONS

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ABSTRACT. Given a Riemannian submersion, we study the relation between lambda constants introduced by G.Perelman on the base manifold and the total space of a Riemannian submersion. We also discuss the relationship between the first eigenvalues of Laplacians on the base manifold and that of the total space. The quantities on warped products are discussed in detail.

Keywords: Riemannian submersion, Eigenvalues, Scalar curvature.

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1. INTRODUCTION

The aim of this paper is to study the relationship between Lambda constants, which will be defined below, the first eigenvalues of Laplacians on the base manifold and that of the total space of a Riemannian submersion. Even though these are classical objects in Differential Geometry and in Mathematical Analysis (see [CM04]), they are missed in literature.

Motivated by Physical background and the Gross Log-Sobolev inequality, G. Perelman [P02] introduced the F-functional

$$F(g, f) = \int_M e^{-f} (|\nabla f|^2 + R) dv_g,$$

where R is the scalar curvature of the Riemannian manifold (M, g) (see [B87]) and dv_g is the volume element of the metric g , with its infimum defined by

$$\lambda(g) = \inf \{ F(g, f) : f \in C_c^\infty(M), \int_M e^{-f} dv_g = 1 \};$$

which will be called the *lambda constant*. It has been showed in [Le07] and [F07] that there is a closed relationship between the Yamabe constants and the lambda constants. Recall that the Yamabe constant $Y(g)$ on (M, g) is the infimum of the action functional

$$A(g) = \int_M R dv_g,$$

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which was introduced by D.Hilbert, among the conformal class $[g]$ with fixed unit volume.

Let us recall some basic notations and concepts about Riemannian submersions from [B87]. Let $p : M \rightarrow B$ be a Riemannian submersion with compact fiber F . Put $F_b = p^{-1}(b)$ for $b \in B$. Given a smooth function u^M on M , we define a smooth function u^B on B by

$$p_*(u^M dv_M) = u^B dv_B.$$

In other word, we have

$$u^B = \int_F u dv_F.$$

Let N be the mean curvature vector field to the fibers F . Let A be the curvature of the horizontal distribution and let T be the second fundamental form of the fibers F . We denote by ∇^M and ∇^B the covariant derivatives on M and on B respectively. We always denote by G^M , G^F , and G^B the same kind of geometrical quantities on M , on F , and on B respectively. For example, Δ^M the Laplacian operator on M . We denote by ∇_{hor} the horizontal part of the covariant derivative on M . That is

$$\nabla_{hor} u = \nabla_{e_\alpha} u e_\alpha$$

where $\{e_\alpha\}$ is an orthonormal basis of $T_{hor}M$ at a point m . We define

$$\delta N(m) = - \sum_{\alpha} \langle \nabla_{e_\alpha}^M N, e_\alpha \rangle.$$

Then we have

$$R^M = R^B + R^F - |A|^2 - |T|^2 - |N|^2 - 2\delta N$$

and

$$\Delta^M u = \nabla_{hor}^2 u + \Delta^F u - \langle \nabla_{hor} u, N \rangle.$$

When $M = B \times F$ with the warped metric $g = g^B + e^{2f} g^F$, where $f \in C^\infty(B)$, we have

$$\Delta^M u = \Delta^B u + e^{-2f} \Delta_{g^F} u + k \langle \nabla^B u, \nabla^B f \rangle.$$

where $k = \dim(F)$.

Let

$$A_0 = \frac{1}{2} \inf_M \left(- \sum \nabla_{e_\alpha} \langle N, e_\alpha \rangle \right)$$

and

$$A_1 = \frac{1}{2} \sup_M \left(- \sum \nabla_{e_\alpha} \langle N, e_\alpha \rangle \right).$$

For the comparison of first eigenvalues on the base and total space, we have the following result.

Theorem 1.1. *Given a compact Riemannian submersion (M, g) . Then we have the following inequality*

$$\lambda_1^M \leq \lambda_1^B + \bar{\lambda}_F + A_1.$$

Here

$$\bar{\lambda}_F = \sup_{b \in B} \lambda_1^F(b).$$

Furthermore, when (M, g) is a warped product, we also have

$$\lambda_1^B + A_0 \leq \lambda_1^M.$$

Since the Euler-Lagrange operator of the F-functional $F(g, f)$ is of the same type of the Laplace operator, we can derive the following result

Theorem 1.2. *Suppose (M, g^M) is a warped product of (B, \check{g}) and (F, g_0) , $g^M = \check{g} + f^2 g_0$, where f is a function on B . If the scalar curvature \hat{R}_0 of the fiber (F, g_0) is bounded from below such that $\hat{R}_0 \geq r_0$, then we have*

$$\lambda^M \geq \lambda^B + c,$$

where $c = \inf(2p \frac{\Delta^B f}{f} + p(7-p) \frac{|df|^2}{f^2} + \frac{1}{f^2} r_0)$.

One may ask if the similar result is true for W-functional. Recall the quantity in question is $\mu(g, \tau)$, which is defined by

$$\mu(g, \tau) = \inf \{ W(g, f, \tau) | f \in C_c^\infty, \tau > 0, \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M e^{-f} dv_g = 1 \},$$

where $W(g, f, \tau)$ is Perelman's W functional introduced in [P02]. We recall that W-functional is defined by

$$W(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M [\tau(4|\nabla u|^2 + Ru^2) - 2u^2 \log u - nu^2] dv_g$$

with $u = e^{-f/2}$, where

$$\frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M e^{-f} dv_g = 1.$$

It seems to us that a similar result should be true, but we can not prove it yet.

On the spin manifold, we have the following result, which related the first eigenvalue of Dirac operator to Perelman's lambda constant.

Theorem 1.3. *Given a compact spin manifold with its Dirac operator D . Let λ_1^D be the first eigenvalue of D in the sense that the absolute value of λ_1^D is the minimum of all eigenvalues of D . Then we have*

$$\lambda_1^{D^2} \geq \lambda(g).$$

Along the Ricci flow, L.Ma [Ma06] obtained a interesting monotonicity formula for eigenvalues of the Laplacian operators.

Here is the plan of the paper. in section two, we introduce some standard formulae. We prove Theorem 1.1 in section three. Theorem 1.2 is proved in section four and Theorem ?? is proved in section five. In the last section, we discuss the relation between the first eigenvalue of the Dirac operator and λ constant of the $F(g, f)$ functional on spin manifolds.

2. PRELIMINARY

In this section we will discuss the relationship between the Laplace operator between the warped product space and its fibre. The material below is standard (see [Lo03]).

Lemma 2.1. *Suppose that a Riemannian submersion $p : M \rightarrow B$ has compact fibre F . For a smooth function ϕ^M on M , we let $\phi^B = \int_F \phi^M dv^F$. Then we have the following identity*

$$\Delta^B \phi^B = \int_F \nabla^2 \phi^M + [\delta N + |\frac{\nabla_{hor} \phi^M}{\phi^M} - N|^2 - |\frac{\nabla_{hor} \phi^M}{\phi^M}|^2] \phi^M dv^F$$

When the Riemannian submersion is a warped product $M = B \times_f F$, $g^M = g^B + f^2 g^F$, we have

$$\Delta^B \phi^B = \int_F \nabla_{hor}^2 \phi^M dv^F + p[\frac{\Delta^B f}{f} + \frac{|df|^2}{f^2}] \phi^B - |N|^2 \phi^B - 2N \phi^B.$$

Proof. The first identity is from [Lo06]. Assume now that M is a warped product. From [B87], we know that

$$N = -\frac{p}{f} \nabla_{hor} f,$$

and

$$\delta N = p[\frac{\Delta^B f}{f} + \frac{|df|^2}{f^2}]$$

where $p = \dim F$. Then,

$$\begin{aligned} \Delta^B \phi^B &= \int_F \Delta^B \phi^M + [\delta N + |\frac{\nabla_{hor} \phi^M}{\phi^M} - N|^2 - |\frac{\nabla_{hor} \phi^M}{\phi^M}|^2] \phi^M dv^F \\ &= \int_F \Delta^B \phi^M dv^F + \int_F \delta N \phi^M dv^F + \int_F |N|^2 \phi^M dv^F \\ &\quad - 2 \int_F \langle \nabla_{hor} \phi^M, N \rangle dv^F \end{aligned}$$

Since in the warped product space, δN , $|N|^2$ are constants on the fibre, we have

$$\Delta^B \phi^B = \int_F \Delta^B \phi^M dv^F + (\delta N + |N|^2) \phi^B - 2 \int_F \langle \nabla_{hor} \phi^M, N \rangle dv^F$$

Since we know (see [Lo06]) that

$$N \phi^B = \int_F (\langle \nabla_{hor} \phi^M, N \rangle - |N|^2 \phi^M) dv^F,$$

we have

$$\begin{aligned} \Delta^B \phi^B &= \int_F \Delta^B \phi^M dv^F + (\delta N - |N|^2) \phi^B - 2N \phi^B \\ &= \int_F \Delta^B \phi^M dv^F + (p \frac{\Delta^B f}{f} + p(1-p) \frac{|df|^2}{f^2}) \phi^B - 2N \phi^B. \end{aligned}$$

□

3. PROOF OF THEOREM 1.1

Let λ_1^M and ϕ^M be the first eigenvalue and the eigenfunction of the Laplacian on M respectively, ie.,

$$\Delta^M \phi^M = -\lambda_1^M \phi^M.$$

Similarly, we define λ_1^B , ϕ^B , λ_1^F , and ϕ^F respectively. Then for $u = \phi^B \phi^F$, we have

$$\Delta^M u = -\lambda_1^B u - \lambda_1^F u - \langle \nabla_{hor} u, N \rangle.$$

Recall that

$$\bar{\lambda}_F = \sup_{b \in B} \lambda_1^F(b).$$

By definition, we have

$$\lambda_1^M = \inf_{\{u \neq 0\}} \frac{\int_M \langle -\Delta^M u, u \rangle}{\int_M u^2}.$$

Hence

$$\lambda_1^M \leq \lambda_1^B + \bar{\lambda}_F + \frac{\int_M \langle \nabla_{hor} u, N \rangle u}{\int_M u^2}.$$

Define

$$A_1 = \frac{1}{2} \sup_M (-\sum \nabla_{e_\alpha} \langle N, e_\alpha \rangle).$$

Then we have

$$\lambda_1^M \leq \lambda_1^B + \bar{\lambda}_F + A_1.$$

This is the upper bound for λ_1^M .

We now give a lower bound for λ_1^M . Note that

$$-\Delta^M \phi^M = \lambda_1^M \phi^M.$$

For simplicity, we let $u = \phi^M$. Then we have

$$-\lambda_1^M u = \nabla_{hor}^2 u + \Delta^F u - \langle \nabla_{hor} u, N \rangle.$$

Integrating over the fiber F , we have

$$-\lambda_1^M u^B = \int_F \nabla_{hor}^2 u - \int_F \langle \nabla_{hor} u, N \rangle dv^F$$

By Lemma 2.1, we have

$$-\lambda_1^M u^B = \Delta^B u^B - (p \frac{\Delta^B f}{f} + p \frac{|df|^2}{f^2}) \phi^B + N u^B$$

Multiplying both sides of the equation above by u^B and taking integration on B , we get

$$\begin{aligned}
\lambda_1^M \int_B |u^B|^2 dv^B &= \int_B (|\nabla^B u^B|^2 dv^B - \frac{1}{2} \int_B N u_B^2 dv^B \\
&+ \int_B p [\frac{\Delta^B f}{f} + \frac{|df|^2}{f^2}] u_B^2 dv^B \\
&= \int_B |\nabla^B u^B|^2 dv^B - \frac{1}{2} \int_B \langle \nabla_{hor} u_B^2, -\frac{p}{f} \nabla_{hor} f \rangle dv^B \\
&+ \int_B p [\frac{\Delta^B f}{f} + \frac{|df|^2}{f^2}] u_B^2 dv^B \\
&= \int_B |\nabla^B u^B|^2 dv^B - \frac{p}{2} \int_B u_B^2 (\frac{\Delta^B f}{f} - \frac{|\nabla f|^2}{f^2}) \\
&+ \int_B p [\frac{\Delta^B f}{f} + \frac{|df|^2}{f^2}] u_B^2 dv^B \\
&= \int_B |\nabla^B u^B|^2 dv^B + \int_B (\frac{p}{2} \frac{\Delta^B f}{f} + \frac{3p}{2} \frac{|df|^2}{f^2}) u_B^2 dv^B.
\end{aligned}$$

Let

$$c = \inf (\frac{p}{2} \frac{\Delta^B f}{f} + \frac{3p}{2} \frac{|df|^2}{f^2})$$

Then we have

$$\lambda_1^M \geq \lambda_1^B + c.$$

4. PROOF OF THEOREM 1.2

We shall use the same spirit of previous section to get bounds for lambda constants.

Proof. Suppose ϕ^M is the first eigenfunction of the F-functional $F(g, f)$. Then

$$\lambda_1^M \phi^M = -4\Delta^M \phi^M + R^M \phi^M.$$

Using the the same procedure in last section, we first do integration in the fibre F .

$$\lambda_1^M \phi^B = -4 \int_F \Delta^M \phi^M dv^F + \int_F R^M \phi^M dv^F$$

where $\phi^B = \int_F \phi^F dv^F$. From [B87], we know

$$R^M = R^F + R^B - p(p-1) \frac{|df|^2}{f^2}.$$

Since the eigenfunction is positive, we have

$$\begin{aligned}
\int_F R^M \phi^M dv^F &= \int_F (R^F + R^B - p(p-1) \frac{|df|^2}{f^2}) \phi^M dv^F \\
&= (R^B - p(p-1) \frac{|df|^2}{f^2}) \phi^B + \int_F R^F \phi^M dv^F \\
&\geq (R^B - p(p-1) \frac{|df|^2}{f^2}) \phi^B + \frac{1}{f^2} R_0 \phi^B.
\end{aligned}$$

Then, we multiply both sides by ϕ^B , and integrate the result on B . Using the result of the last theorem, we have

$$\begin{aligned}
\lambda_1^M \int_B (\phi^B)^2 &= -4 \int_B \phi^B (\int_F \Delta^M \phi^M dv^F) dv^B + \int_B \phi^B \int_F R^M \phi^M dv^F dv^B \\
&= 4 \int_B |\nabla^B \phi^B|^2 dv^B + 4 \int_B (\frac{p}{2} \frac{\Delta^B f}{f} + \frac{3p}{2} \frac{|df|^2}{f^2}) \phi_B^2 dv^B \\
&\quad + \int_B \phi^B \int_F R^B + R^F - p(p-1) \frac{|df|^2}{f^2}) \phi^M dv^F dv^B. \\
&= \int_B 4 |\nabla^B \phi^B|^2 + R^B (\phi^B)^2 dv^B \\
&\quad + \int_B (2p \frac{\Delta^B f}{f} + p(7-p) \frac{|df|^2}{f^2}) (\phi^B)^2 dv^B \\
&\quad + \int_B \phi^B \int_F \frac{1}{f^2} R_0 \phi^M dv^F dv^B \\
&\geq \lambda_1^B \int_B (\phi^B)^2 dv^B + (c + \frac{1}{f^2} r_0) \int_B (\phi^B)^2 dv^B,
\end{aligned}$$

where we have used the notation

$$c = \inf (2p \frac{\Delta^B f}{f} + p(7-p) \frac{|df|^2}{f^2}).$$

□

5. RELATED CONSTANTS

In this section, we shall discuss the relation between the first eigenvalue of the Dirac functional and the lambda constants on spin manifolds.

We now prove Theorem 1.3.

Proof. We consider the Dirac functional

$$I(u) = \int_M |Du|^2$$

on the spin manifold (M, g) , where u is the spinor field on M . Using the Lichnorowicz formula

$$D^2 = \nabla^2 + \frac{R}{4},$$

we have

$$I(u) = \int_M (|\nabla u|^2 + \frac{R}{4}|u|^2)$$

which is very similar to Perelman's F-functional. In fact, using Kato's inequality

$$|\nabla u| \geq |d|u||,$$

we have

$$I(u) \geq J(g, f) \geq \lambda(g)$$

for $e^{-f} = |u|^2$. Let λ_1^D be the first eigenvalue of Dirac operator D , i.e.,

$$Du_D = \lambda_1^D u_D.$$

Then we have

$$\lambda_1^{D^2} = I(u_D) \geq \lambda(g).$$

□

By this inequality, we can get a nice lower bound for $\lambda_1^{D^2}$ by studying $\lambda(g)$. We believe that this kind of idea should be useful in the study of Dirac operators.

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